# Geometric Algebra - Complex Numbers Without $\sqrt{-1}$.

Russell Goyder, January 2023.

References:


[GLD] Imaginary Numbers are not Real – the Geometric Algebra of Spacetime - Gull, Lasenby and Doran

[DL] Geometric Algebra for Physicists - Doran and Lasenby


Disclosure: Anthony Lasenby was my PhD advisor.

Overview:

# Clifford's Axioms [OL 4.1]

Special case of Clifford Algebra $CL(V, Q)$ for a vector space $V$ over a field $k$, equipped with a quadratic form $Q : V \to k$.

Let the field be $\mathbb{R}$ and the quadratic form $Q$ be provided by a metric $g$ with signature $(p, q)$. Then denote the geometric algebra in $n = p + q$ dimensions by

$$G(p,q) = CL(V, g).$$

and the quadratic form by $\underbrace{a \cdot b}_{\text{scalar, or inner product.}} = g(a,b)$ for $a, b \in V$.

$a \cdot b \in \mathbb{R}$.

In addition to the inner product, Clifford gives us the <u>geometric product</u> $ab$, for $a, b \in V$, with axioms:

1. Closure : $ab \in G(p,q)$
2. Identity : $\exists\ \mathbb{1} \in G(p,q)$ s.t. $\mathbb{1} a = a \mathbb{1} = a \in G(p,q)$
3. Associativity : $a(bc) = (ab)c$
4. Distributivity : $a(b+c) = ab + ac$

Note : no commutativity

5. Relation to inner product : $a^2 = a \cdot a = g(a,b) \in \mathbb{R}$
ie : square of any vector is a scalar.

$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2$$

$$\therefore \; ab + ba = \underbrace{(a+b)^2 - a^2 - b^2}_{\text{all } \in \mathbb{R} \text{ by axiom 5}}$$

So, define the inner product of two vectors as

$$a \cdot b = \underbrace{\frac{1}{2}(ab + ba)}_{\text{symmetric part of geometric product}}$$

Denote antisymmetric part of the geometric product by $\wedge$ :

exterior product

$$a \wedge b = \frac{1}{2}(ab - ab)$$

So that $ab = \underline{a \cdot b} + \underline{a \wedge b}$

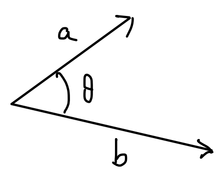
Vectors $a, b$ have "grade" 1. Scalars have grade 0.
What about this?

$a \wedge b$ is a "bivector", of grade 2.

It represents an oriented area in the plane containing $a$ and $b$.

$$a \wedge b = - b \wedge a$$

$|a|\,|b| \sin\theta$ where $|a| = \sqrt{a^2}$ and
$|b| = \sqrt{b^2}$



Note the similarity to the cross product of two vectors, $a \times b$, which only exists in 3 dimensions, whereas bivectors exist in any dims $\geq 2$, as do higher grade objects:

$$a_1 \wedge a_2 \wedge \ldots \wedge a_r = \frac{1}{r!} \sum_{\text{all perms}} (-1)^{\varepsilon} a_{k_1} a_{k_2} \cdots a_{k_r}$$

$\varepsilon$ is $+1$ for even perms and $-1$ for odd perms,

## $G(2,0)$ - the geometric algebra of the plane [DL 2.3.2]

Span 2-d Euclidean space with an orthonormal basis $\{e_1, e_2\}$. Then, an arbitrary element of $G(2,0)$ can be written:

$$\underset{\text{"multivector"}}{A} = \underbrace{a_0}_{\text{scalar}} + \underbrace{a_1 e_1 + a_2 e_2}_{\text{vector}} + \underbrace{a_3 e_1 \wedge e_2}_{\text{bivector}} \quad \text{for } a_0, \ldots a_3 \in \mathbb{R}$$

$e_1^{\,2} = 1$

$e_1 \wedge e_2 = e_1 e_2$ because $e_1 \cdot e_2 = 0$

$$AB = (a_0 + a_1 e_1 + a_2 e_2 + a_3 e_1 e_2)(b_0 + b_1 e_1 + b_2 e_2 + b_3 e_1 e_2)$$

$$\begin{aligned}
&= a_0 b_0 + a_1 b_1 + a_2 b_2 - a_3 b_3 \\
&+ (a_0 b_1 + b_0 a_1 + a_3 b_2 - a_2 b_3)\, e_1 \\
&+ (a_0 b_2 + a_2 b_2 + a_1 b_3 - a_3 b_1)\, e_2 \\
&+ (a_0 b_3 + b_0 a_3 + a_1 b_2 - a_2 b_1)\, e_1 \wedge e_2
\end{aligned}$$

$$\begin{aligned}
a_3 e_1 e_2 b_3 e_1 e_2 &= a_3 b_3\, e_1 e_2 e_1 e_2 \\
&= a_3 b_3 (-e_1 \underbrace{e_2 e_2}_{1} e_1) \\
&= -a_3 b_3
\end{aligned}$$

$$\begin{aligned}
a_2 e_2 b_3 e_1 e_2 &= a_2 b_3\, e_2 e_1 e_2 \\
&= a_2 b_3 (-e_1 e_2^{\,2}) \\
&= -a_2 b_3 e_1
\end{aligned}$$

## Complex numbers [DL 2.3.3]

The algebra of complex numbers is isomorphic to the even-grade sub-algebra of $G\ell(2,0)$:

$$\underbrace{Z}_{\text{even-grade multivector}} = \underbrace{x}_{\text{scalar}} + y\underbrace{e_1 e_2}_{\text{bivector}} = x + \underbrace{Iy}_{\text{notation}} \quad \text{where } I = e_1 e_2$$

$x, y \in \mathbb{R}$

$$I^2 = e_1 e_2 e_1 e_2 = -e_1 \underbrace{e_2 e_2}_{e^2 = 1} e_1 = -e_1^2 = 1$$

So $I = \sqrt{-1}$ ?

Sort of, but not really - more about this on the next page.

A complex number $z \in \mathbb{C}$, $z = x + iy$, $i = \sqrt{-1}$ defines a point in the complex plane whose Cartesian coodinates are $(x, y)$, $x, y \in \mathbb{R}$.

But the natural specification of a point in a plane is its position vector: $w = xe_1 + ye_2$.

Clearly, $Z = e_1 w$

This is the fundamental object in complex analysis

This is the fundamental object in geometry. It is independent of choice of basis / coordinates.

This vector picks out a special direction for the real axis

# $\sqrt{-1}$

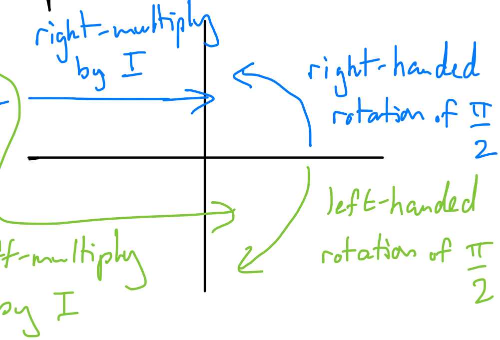
The usual story goes: there is no real number equal to $\sqrt{-1}$, but equations where squares take negative values arise in many problems, so let's invent something mysterious called an "imaginary" number $i$ and set $i = \sqrt{-1}$. Then $z = x + iy$ is a point in a plane, and through complex analysis we can solve problems that vector analysis cannot.

(well, exterior)

Geometric algebra says: vector analysis is incomplete because it has not unified the scalar and (bi-)vector products into a single geometric product, and if you do, there is no need for the mystery. $I = e_1 e_2$ has a natural geometric interpretation – as a bivector encoding the orientation of the plane and acting as a rotation operator on vectors:

$$
\begin{aligned}
I e_1 &= e_1 e_2 e_1 = -e_2 \\
e_1 I &= e_1 e_1 e_2 = e_2 \\
I e_2 &= e_1 e_2 e_2 = e_1 \\
e_2 I &= e_2 e_1 e_2 = -e_1
\end{aligned}
$$

in n-dim space, $n \geq 2$




Clearly, rotating twice by $\frac{\pi}{2}$ in either direction results in the opposite direction (rotation by $\pi$):

ie minus sign

$$I^2 \mathbb{1} = -\mathbb{1}.$$

## Quaternions [DL 1.4, 2.4.2]

In 1843, Hamilton generalized the complex numbers to 3d by adding two more square roots of minus 1:

$$i^2 = j^2 = k^2 = ijk = -1$$

With this additional property, "3d complex numbers":

$$t + xi + yj + zk \qquad t, x, y, z \in \mathbb{R}$$

form a closed algebra. However, complex numbers are already "2d", so now we have 6d? And there is no intrinsic notion of which plane in 3d space $i$, $j$ or $k$ belong to.

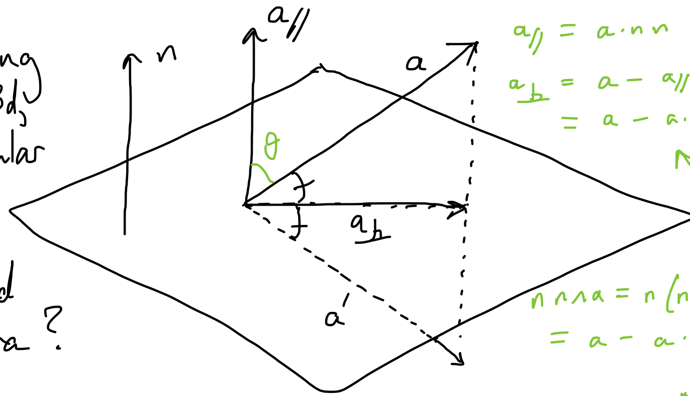
However, Hamilton's quaternions are naturally embedded in $G(3,0)$:

$$i \longleftrightarrow e_1 e_2$$
$$j \longleftrightarrow -e_2 e_3$$
$$k \longleftrightarrow e_3 e_1$$

Quaternions are a left-handed set of bivectors, whereas $i$, $j$ and $k$ were chosen to be a right-handed set of vectors.

# Complex Conjugation and Reflection [DL 2.3.3]

Reflect vector $a$ along unit vector $n$ (in 3d, in the plane perpendicular to $n$), to get $a'$.




$a_{/\!/} = a\cdot n\, n$

$a_{\perp} = a - a_{/\!/}$
$= a - a\cdot n\, n$

$n\, n\wedge a = n\,(na - n\cdot a)$
$= a - a\cdot n\, n$

How is this expressed in geometric algebra?

Resolve $a$ into components $/\!/$ and $\perp$ to $n$:

$$
\begin{aligned}
a &= n^2 a \\
&= n\, na \\
&= \underbrace{n\, n\cdot a}_{a_{/\!/}} + \underbrace{n\, n\wedge a}_{a_{\perp}}
\end{aligned}
$$

convention: this before this

(grade 3 component vanishes because $n\wedge n\wedge a$ antisymm)

$$
\begin{aligned}
a' = a_{\perp} - a_{/\!/} &= n\, n\wedge a - n\, n\cdot a \\
&= -n\,(a\wedge n + a\cdot n) \\
&= -nan
\end{aligned}
$$

Ie $\boxed{a' = -nan}$

This concise formula applies in any dimension, and to multivectors of all grades, not just vectors in 3d.

The complex conjugate of $z$, $z^* = x - iy$.

GA equiv is

$$x - Iy = x - e_1 e_2 y$$
$$= (x e_1 + y e_2) e_1$$
$$= w e_1$$
$$= \tilde{z}$$

where $\sim$ indicates "reversion" - reverse the order of all factors in the geometric product.

In terms of the vector $w$ directly, complex conjugation is equivalent to a reflection in the $e_1$ axis. From the previous page,

$$-e_2 w e_2 = -e_2 (x e_1 + y e_2) e_2$$
$$= -x e_2 e_1 e_2 - y e_2 e_2 e_2$$
$$= x e_1 - y e_2 \quad \text{as expected.}$$

Building on reflections, rotations are expressed particularly elegantly in geometric algebras of arbitrary dimension $n \geq 2$.

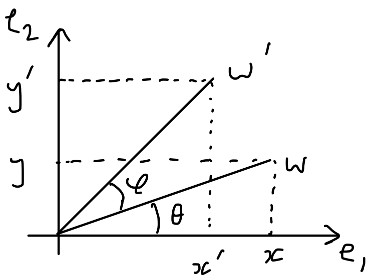
# Rotations [DL 2.3.4]

On page 5 we saw that right-multiplication by $I$ causes a right-handed rotation by $\frac{\pi}{2}$. For an arbitrary angle $\varphi$, we have:

(power series still work!)

$$e^{I\varphi} = \sum_{n=0}^{\infty} \frac{(I\varphi)^n}{n!} = \cos\varphi + I\sin\varphi$$

Apply to vector $w$:




$$\begin{aligned}
w' &= w e^{I\varphi} \\
&= (\cos\theta\, e_1 + \sin\theta\, e_2)(\cos\varphi + I\sin\varphi) \\
&= \cos\theta\cos\varphi\, e_1 + \cos\theta\sin\varphi\, \underbrace{e_1 I}_{e_1e_1e_2 = e_2} \\
&\quad + \sin\theta\cos\varphi\, e_2 + \sin\theta\sin\varphi\, \underbrace{e_2 I}_{e_2e_1e_2 = -e_1} \\
&= (\cos\theta\cos\varphi - \sin\theta\sin\varphi)\, e_1 + (\sin\theta\cos\varphi + \cos\theta\sin\varphi)\, e_2
\end{aligned}$$

So $\quad w' = \cos(\theta+\varphi)\, e_1 + \sin(\theta+\varphi)\, e_2$

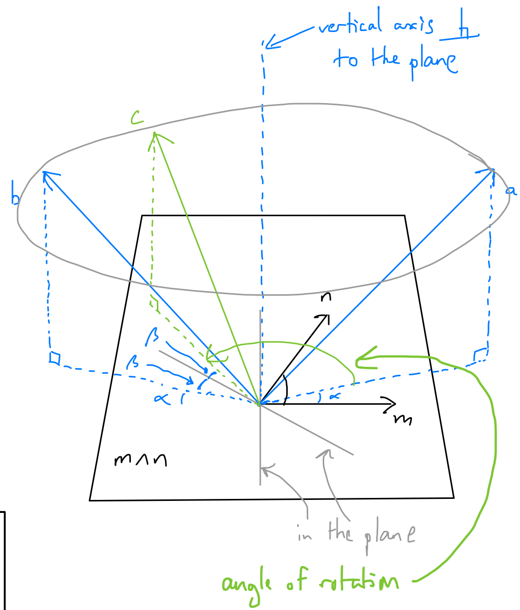
Note how two separate roles of complex numbers are separated here:

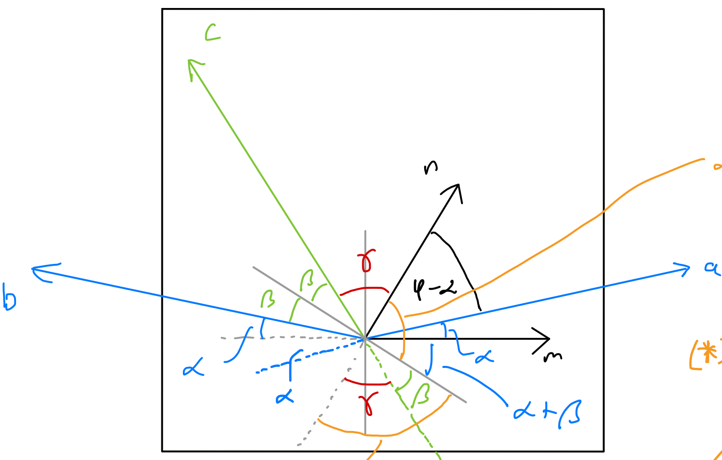
grade 1 → 1. position vector of a point in 2-d (Euclidean) space
2. transformations of those points (even-grade multivector)

These roles generalize differently in higher dimensions.

# Rotors [DL 2.7]

Express an arbitrary rotation with two reflections, in the planes perpendicular to unit vectors $m$ and $n$. Vector $a$ is first reflected by $m$ to form vector $b$, then reflected by $n$ to form the rotated vector $c$.




$$\alpha + \beta + \cancel{\alpha} + \varphi - \cancel{\alpha} = \frac{\pi}{2}$$

$$\therefore \alpha + \beta + \varphi = \frac{\pi}{2} \quad \text{— } (**)$$

$$\gamma + \beta = \frac{\pi}{2} \quad \text{— } (*)$$

$(*) - (**)$:

$$\gamma = \alpha + \varphi$$

Angle of rotation (between vectors $a$ and $c$) is: $\gamma + \varphi - \alpha = \alpha + \varphi + \varphi - \alpha = \boxed{2\varphi}$

where $\varphi$ is the angle between $m$ and $n$, $m \cdot n = \cos\varphi$.

From page 7 $\quad b = -mam$
and $\quad c = -nbn = nmamn$

Define $R_{nm} = nm$, then rotations are achieved via:

$$c = R_{nm}\, a\, \tilde{R}_{nm}$$ ← reversion, from page 8.

But on pages 5 and 9 we had rotations in the $e_1e_2$ plane being performed by right-multiplication of $e^{I\varphi}$: $w' = we^{I\varphi}$. What's going on? Two issues to resolve:

1: $e^{I\varphi}$ vs $mn = R_{nm}$, and 2: $w' = we^{I\varphi}$ vs $w' = R_{nm}\, w\, \tilde{R}_{nm}$

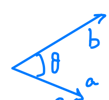
$$\begin{aligned} R_{nm} = nm &= n\cdot m + n\wedge m \\ &\phantom{=}\ \cos\varphi + n\wedge m \end{aligned}$$

← this is not a unit bivector like $I$

For any vectors $a$ and $b$:

$ab = a\cdot b + a\wedge b$
$ba = a\cdot b - a\wedge b$

$$\begin{aligned} (a\wedge b)(a\wedge b) &= (ab - a\cdot b)(a\cdot b - ba) \\ &= -a\underbrace{b^2}_{\text{scalar}}a - (a\cdot b)^2 + (a\cdot b)\underbrace{(ab+ba)}_{2a\cdot b} \\ &= -a^2b^2 + 2(a\cdot b)^2 \\ &= -a^2b^2(1-\cos^2\theta) \\ \therefore\quad (a\wedge b)^2 &= -a^2b^2\sin^2\theta \end{aligned}$$




$a\cdot b = |a||b|\cos\theta$

$m$ and $n$ are unit vectors, so $(m\wedge n)^2 = -\sin^2\varphi$

not a unit bivector, because $\varphi \neq \frac{\pi}{2}$

Can construct a unit bivector in the $m \wedge n$ plane:

$$B = \frac{m \wedge n}{\sin \varphi} \qquad B^2 = -1$$

parallel and same handedness

If the $m \wedge n$ plane is the $e_1 \wedge e_2$ plane, then $B = I$. Let's assume that to make contact with the previous discussion on complex numbers. So, the rotor $R_{nm}$

$$\begin{aligned} R_{nm} &= \cos \varphi + m \wedge n \\ &= \cos \varphi - I \sin \varphi = e^{-I\varphi}. \end{aligned}$$

Recall from page (10) that $w' = R_{nm} w \tilde{R}_{nm}$ is rotated in the $n \wedge m$ plane by angle of $\underline{2\varphi}$, and so the rotor for an angle of $\varphi$ is:

$$R_\varphi = e^{-I\varphi/2}, \qquad \tilde{R}_\varphi = e^{I\varphi/2}, \qquad w' = R_\varphi w \tilde{R}_\varphi.$$

Now, $e_1$ and $e_2$ both anticommute with $I = e_1 e_2$, so

$$R_\varphi w = (\cos \varphi - I \sin \varphi)(x e_1 + y e_2) = w R_{-\varphi} = w \tilde{R}_\varphi$$

$$\text{So } w' = w \tilde{R}_\varphi^{\,2} = w R_{2\varphi} = w e^{I\varphi} = e^{-I\varphi} w$$

Ie, we can write a rotation as either the two-sided half-angle expression or the one-sided full-angle expression in 2d. Only one of them generalizes to n-dims, however.

Consider 3-d : add $e_3$, a third orthonormal basis vector. Any rotation in the $e_1 e_2$ plane should leave it untouched because $e_3$ is the axis of rotation.

But $e_3$ <u>commutes</u> with $I$ :

$$e_3 I = e_3 e_1 e_2 = -e_1 e_3 e_2 = e_1 e_2 e_3 = I e_3$$

And so while $R_\varphi e_3 \tilde{R}_\varphi = \underbrace{R_\varphi \tilde{R}_\varphi}_{nmmn \,=\, n^2 m^2 = 1} e_3 = e_3$,

$$e_3 R_{2\varphi} = e_3 e^{I\varphi} \neq e_3 .$$

Only the two-sided rotation law generalizes to n-dimensions.

<u>Notes :</u>

There is only an <u>axis</u> of rotation in 3 dimensions. It is better to think of rotations as happening in a plane - the one encoded by the bivector in the rotor.

It turns out that the same rotation law applies not just to vectors, but all multivectors, no matter the dimension. [DL 4.2.1]

We have discussed complex arithmetic, but it turns out there are rich implications of geometric algebra for complex analysis; holo/meromorphic functions and calculus on them. [DL 6]